Polyominoes on Twisted Cylinders

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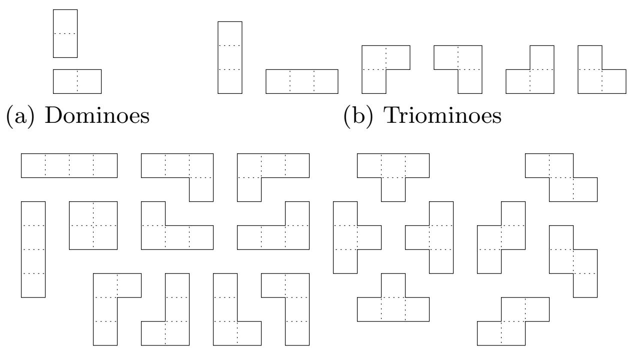

(a) Dominoes (b) Triominoes (c) Tetrominoes

Figure 1: All fixed dominoes, triominoes, and tetrominoes in the plane.

ABSTRACT

In this video we show how to enumerate polyominoes on twisted cylinders, and explain how to use them for setting lower bounds on the asymptotic growth rate of polyominoes in the plane.

Categories and Subject Descriptors

G.2.1 [**Mathematics of Computing**]: Discrete Mathematics—*Combinatorics*

Keywords

Polyominoes; polycubes

1. INTRODUCTION

A polyomino of size n is an edge-connected set of n cells on the square lattice $\mathbb{Z}^2$. In statistical physics, polyominoes and their higher-dimensional generalizations (polycubes) play an important role in computing the mean cluster density in percolation processes, such as fluid flow in random media [3], and in modeling the collapse of branched polymer molecules in dilute solution [10]. Two *fixed* polyominoes are considered distinct if they differ in shape *or* orientation. Figure 1 shows all the fixed polyominoes of sizes two, three, and four. The number of fixed polyominoes of size n is usually denoted by $A(n)$. There are two long-standing open problems related to the study of polyominoes.

1. The enumeration of polyominoes, that is, finding a formula for $A(n)$ or computing $A(n)$ for specific values of n; and

2. Computing the quantity $\lim_{n\to\infty} \sqrt[n]{A(n)}$, the asymptotic *growth rate* of polyominoes (also known as "Klarner's constant").

To date, no formula is known for $A(n)$. Redelmeier [8] introduced the first efficient algorithm for counting polyominoes, in the sense that it generated all polyominoes sequentially *without* repetitions. The best known method (in terms of running time) for counting fixed polyominoes is a transfer-matrix algorithm suggested by Jensen in 2001. By running a parallel version of his algorithm, Jensen [5] computed $A(n)$ up to $n = 56$.

Klarner [6] showed in a seminal work in 1967 that the limit $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$ exists. Only three decades later did Madras [9] show that $\lim_{n\to\infty} A(n+1)/A(n)$ also exists and, hence, is equal to λ. At the present time not even a single significant decimal digit of λ is known. The best-known lower [2] and upper [7] bounds on λ are 3.9801 and 4.6496, respectively. It is generally assumed (see, e.g., [4]), as a conclusion from numerical methods applied to the known values of $A(n)$, that $\lambda \approx 4.06 \pm 0.02$. Jensen [5] refined this analysis, estimating λ at 4.0625696 ± 0.0000005.

2. TWISTED CYLINDERS

Twisted cylinders were introduced in the context of polyominoes in [2]. For a fixed natural number w, the twisted cylinder of *width* (or perimeter) w is the surface obtained from the Euclidean plane by identifying all pairs of points of the form (x, y), $(x - kw, y + k)$, for $k \in \mathbb{Z}$. Figure 2 shows a twisted cylinder of width 3 and a polyomino of size 9 embedded on it.

In [2], twisted cylinders were used in order to set a lower bound on λ. It was shown that for any $w \geq 1$, the constant λ_w, the asymptotic growth rate of polyominoes on a twisted cylinder of width w, is a lower bound on λ. It was also shown that the sequence (λ_w) is monotone increasing. Elements of the sequence were computed up to $\lambda_{22} = 3.9801\ldots$, setting this bound on λ.[1] It was also proven [1] that the sequence (λ_w) converges to λ.

[1]Recently, we computed $\lambda_{23} = 3.9856\ldots$, thereby improved the lower bound on λ.

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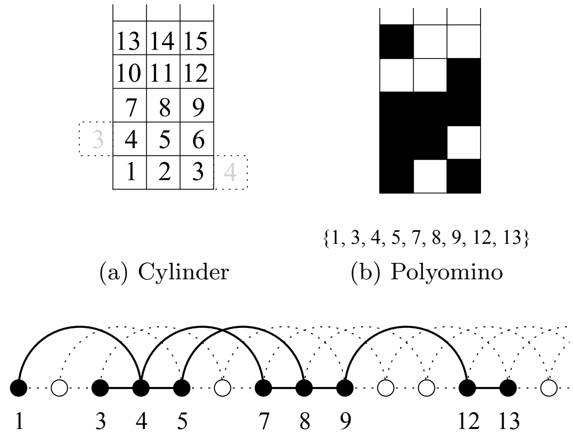

(a) Cylinder (b) Polyomino

(c) Subgraph of the host graph

Figure 2: A polyomino on the twisted cylinder of width 3.

One can find in [1] formulae for the number of polyominoes on twisted cylinders obtained by building finite automata that model the "growth" of polyominoes on these cylinders, computing the transfer matrices of these automata, and deducing from these matrices the generating functions for the sequences that enumerate polyominoes on the twisted cylinders. This method proves that the formula for the number of polyominoes on a twisted cylinder of *any* width obeys a linear recurrence. In practice, the size of the involved matrix grows exponentially with w (as do Motzkin numbers), and so, the amount of computations, as well as the orders of the computed formulae, becomes prohibitively large for large values of w. For example, for $w = 10$, the recurrence includes 2,168 terms, with the largest coefficient being about $6.39 \cdot 10^{129}$ (432 bits).

3. THE VIDEO

The video shows the relation between polyominoes and twisted cylinders. First, it defines what polyominoes are, specifies possible uses of them, and mentions the best known lower and upper bounds on the asymptotic growth rate of polyominoes in the plane. It then illustrates the process of counting polyominoes on twisted cylinders by running the algorithm on a twisted cylinder of width 3. It also demonstrates how this process can be modeled by a finite automaton. Finally, the video shows how we compute the growth rate of polyominoes on twisted cylinders, and concludes with how this implies lower bounds on the respective rate in the plane. Figure 3 shows two snapshots from the video.

The video was produced on a 2.53GHz DELL 64 processor PC with 4GB of RAM. The scenes and animations were designed using the Autodesk Maya 2013 (student version) modeling software. The video was constructed using Windows Live Movie Maker.

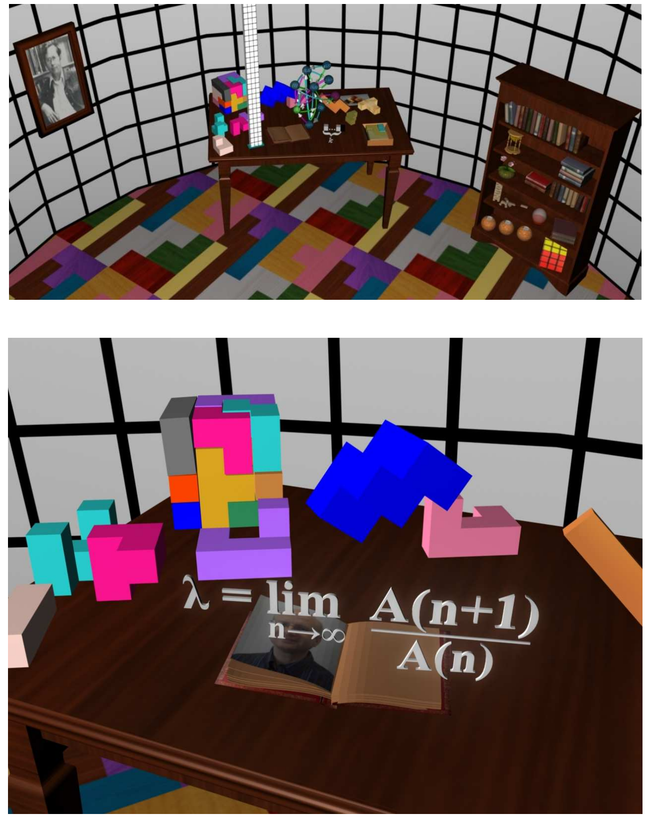

Figure 3: Snapshots from the video.

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